Abstract—We consider a problem of multi-path routing in a communication network, where flow control is implemented with a prescribed number of routes available for each source/destination pair. A classical approach is to formulate this as a network optimization problem where decentralized update rules converge to its optimal solution. It has been reported in the literature that the problem is in this case not strictly convex leading to oscillatory behaviours when classical primal/dual dynamics are implemented. In this paper we show that the use of appropriate higher order dynamics, which are fully localized, can provide guarantees for convergence to the optimal solution. Furthermore, we show that the instability observed in the unmodified scheme can be severe, leading to unbounded behaviour with arbitrarily small noise perturbations. The results are also illustrated with simulations showing how these modifications can lead also to improved performance.

I. INTRODUCTION

Multipath routing is a problem that has received considerable attention within the communications literature due to the significant advantages it can provide relative to congestion control algorithms that use single paths [11]. Nevertheless their implementation is not directly obvious as the availability of multiple routes can render the network prone to route flapping instabilities [19].

A classical approach to analyze such algorithms is to formulate them as solving a network optimization problem where aggregate user utilities are maximized subject to capacity constraints [9]. In the seminal work in [9] it was noted that when capacity constraints are relaxed with penalty functions and primal algorithms are considered, then convergence can be guaranteed despite the presence of multiple routes. In order, however, to achieve the network capacities, dual or primal/dual algorithms need to be deployed [17]. Nevertheless, when multiple routes per source/destination pair are available the corresponding optimization problem is known to be not strictly convex and the use of classical gradient dynamics can lead to unstable behaviour [18], [8], [1]. In order to address this issue various studies have considered relaxations that lead to a modified optimization problem that is strictly convex [18], [4]. This leads to algorithms with guaranteed convergence, but with the equilibrium solution deviating from that of the solution of the original optimization problem.

In this paper we consider a multi-path routing problem with a fixed number of routes per source/destination pair, as in [9], [18], [12], [13]. For such schemes we investigate algorithms that allow the corresponding network optimization problem to be solved without requiring any relaxation in its solution or any additional information exchange. In particular, we show that this is feasible by incorporating appropriate higher order dynamics in the local update rules. It should be noted the lack of strict convexity in conjunction with the hybrid nature of the dynamics render the analysis more involved than that of classical gradient dynamics. One of the contributions of the paper is to provide a proof of the convergence of the proposed algorithm with the switching behaviour of the underlying system explicitly taken into account. Furthermore, we show that the instability observed when unmodified gradient dynamics are used can lead to unbounded behaviour with arbitrarily small noise perturbations, despite the nonlinearity in the dynamics.

The paper is structured as follows. We start with various preliminaries in Section II. The problem is then formulated in Section III. The instability of classical gradient schemes and their susceptibility to noise amplification is discussed in subsection III-B. The stability of modified schemes is then shown in Section IV. Numerical examples are given in Section V and finally conclusions are drawn in Section VI. Due to page constraints the proof of Proposition I has been omitted and may be found in an extended version of this manuscript [7].

II. PRELIMINARIES

A. Notation

Real numbers are denoted by $\mathbb{R}$ and non-negative real numbers as $\mathbb{R}_+$. For vectors $x, y \in \mathbb{R}^n$ the inequality $x < y$ holds if the corresponding inequality holds for each pair of components, $d(x, y)$ is the Euclidean metric and $|x|$ denotes the Euclidean norm.

The space of $k$-times continuously differentiable functions is denoted by $C^k$. For a sufficiently differentiable function $f(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ we denote the vector of partial derivatives of $f$ with respect to $x$ as $f_x$, respectively $f_y$. The Hessian matrices of $f$ with respect to $x$ and $y$ are denoted $f_{xx}$ and $f_{yy}$ with $f_{xy}$ and $f_{yx}$ denoting the matrices of mixed partial derivatives in the appropriate arrangement.

For a matrix $A \in \mathbb{R}^{n \times m}$ we denote its kernel and transpose by $\ker(A)$ and $A^T$ respectively. If $A$ is in addition symmetric, we write $A < 0$ if $A$ is negative definite. If $n = m$ we denote the trace of $A$ by $\text{Tr}A$. The diagonal matrix with entries $x_1, x_2, \ldots, x_n$ will be denoted $\text{diag}(x_1, x_2, \ldots, x_n)$.

When we consider a concave-convex function $\varphi(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ (see Definition 1) we shall denote the pair $z = (x, y) \in \mathbb{R}^{n+m}$ in bold, and write $\varphi(z) = \varphi(x, y)$. 

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The full Hessian matrix will then be denoted \( \varphi_{zz} \). Vectors in \( \mathbb{R}^{n+m} \) and matrices acting on them will be denoted in bold font (e.g. \( A \)). Saddle points (see Definition 2) of \( \varphi \) will be denoted \( \bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^{n+m} \). We will also use this notation for equilibrium points.

For \( x \in \mathbb{R}, y \in \mathbb{R}_+ \) we define \( [x]_y^+ = x \) if \( y > 0 \) and \( \max(0, x) \) if \( y = 0 \).

**B. Definitions**

In this section we introduce various notions that will be used throughout the paper.

**Definition 1** (Concave-Convex function). We say that a function \( g(x, y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is (strictly) concave in \( x \) (respectively \( y \)) if for any fixed \( y \) (respectively \( x \)), \( g(x, y) \) is (strictly) concave as a function of \( x \), (respectively \( y \)). If \( g \) is concave in \( x \) and convex in \( y \) we say that \( g \) is concave-convex.

**Definition 2** (Saddle point). For a concave-convex function \( \varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) we say that \( (\bar{x}, \bar{y}) \in \mathbb{R}^{n+m} \) is a saddle point of \( \varphi \) if for all \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \) we have the inequality \( \varphi(x, \bar{y}) \leq \varphi(\bar{x}, \bar{y}) \leq \varphi(\bar{x}, y) \).

If \( \varphi \) is in addition \( C^1 \) then \( (\bar{x}, \bar{y}) \) is a saddle point if and only if \( \varphi_x(\bar{x}, \bar{y}) = 0 \) and \( \varphi_y(\bar{x}, \bar{y}) = 0 \).

**Definition 3** (Gradient method). Given a concave-convex function \( C^2 \ni \varphi : \mathbb{R}^{n+m} \to \mathbb{R} \) we define the gradient method as the solutions of the differential equation

\[
\dot{x} = \varphi_x, \quad \dot{y} = -\varphi_y.
\]

Throughout this paper we consider a subgradient method where some of the variables are constrained to stay non-negative.

**Definition 4** (Subgradient method). Given a concave-convex function \( C^2 \ni \varphi : \mathbb{R}^{n+m} \to \mathbb{R} \) and, sets of indices \( I \subseteq \{1, \ldots, n\} \) and \( J \subseteq \{1, \ldots, m\} \) we define the subgradient method by the differential equation

\[
\dot{x}_i = \begin{cases} [\varphi_{x_i}^+] & \text{for } i \in I \\ \varphi_{x_i} & \text{for } i \notin I \\ \end{cases}, \quad \dot{y}_j = \begin{cases} [\varphi_{y_j}^+] & \text{for } j \in J \\ -\varphi_{y_j} & \text{for } j \notin J \end{cases}
\]

with initial conditions satisfying \( x_i, y_j \geq 0 \) for \( i \in I, j \in J \).

Here we interpret (2) in the Carathéodory sense, i.e. consider absolutely continuous solutions that satisfy (2) for almost all times \( t \geq 0 \).

The existence and uniqueness of trajectories of the subgradient method (2) can be shown by interpreting it as a projected dynamical system (see e.g. [14]). See [2] for a more detailed discussion of this issue.

**III. Multi-path routing**

**A. Problem formulation**

We consider a multi-path routing problem where each source/destination pair has a fixed number of routes.

In particular, we consider a network that consists of sources \( s_1, \ldots, s_m \), routes \( r_1, \ldots, r_n \), and links \( l_1, \ldots, l_l \). Each source \( s_i \) is associated with a unique destination for a message which is to be routed. Every route \( r_j \) has a unique source \( s_i \), and we write \( r_j \sim s_i \) to mean that \( s_i \) is the source associated with route \( r_j \). Routes \( r_j \) each use a number of links, and we write \( r_j \sim l_k \) to mean that the link \( l_k \) is used by the route \( r_j \). The desired running capacity of the link \( l_k \) is denoted \( C_k \), and \( 0 \leq C \in \mathbb{R}^l \) is the vector of these capacities. We let \( A \) be the connectivity matrix, so that \( A_{kj} = 1 \) if \( l_k \sim r_j \) and 0 otherwise. In the same way we set \( H_{ij} = 1 \) if \( s_i \sim r_j \) and 0 otherwise. \( x_j \) denotes the current usage of the route \( r_j \). We associate to each source \( s_i \) a strictly concave, increasing utility function \( U_i \).

The problem of maximising total utility over the network is stated as

\[
\max_{x \geq 0, Ax \leq C} \sum_{s_i} U_i \left( \sum_{r_j \sim s_i} x_j \right).
\]

Here the first sum is over all sources \( s_i \), and the second over routes \( r_j \) with \( r_j \sim s_i \). (We shall use such notation throughout the paper.) This optimisation problem is associated with the Lagrangian

\[
\varphi(x, y) = \sum_{s_i} U_i \left( \sum_{r_j \sim s_i} x_j \right) + y^T (C - Ax).
\]

where \( y \in \mathbb{R}_+^l \) are Lagrange multipliers that relax the \( Ax \leq C \) constraint. A common approach in the context of congestion control is to consider primal-dual dynamics originating from this Lagrangian so as to deduce decentralized algorithms for solving the network optimisation problem [9],[17]. This gives rise to the subgradient method

\[
\dot{x}_j = \left[ U'_i \left( \sum_{s_i \sim r_j} x_k \right) - \sum_{l_k \sim r_j} y_{lk} \right]_{x_j}^+ \quad \dot{y}_{lk} = \left[ \sum_{l_k \sim r_j} x_j - C_k \right]_{y_{lk}}^+.
\]

where \( s_i \sim x_j \) in the equation for \( \dot{x}_j \) and \( U'_i \) is the derivative of the utility function \( U_i \). Note that the eigenvalues of (5) are saddle points of the Lagrangian (under the positivity constraints on \( x \) and \( y \)) and hence also solutions of the optimization problem (4) (Slater’s condition is assumed to hold throughout the paper).

The dynamics (5) are also localised in the sense that the update rules for \( x_j \) depend only on the current usage, \( x_k \), of routes with the same source and of the congestion signals associated with links on these routes. In the same way the update rules for congestion signals \( y_{lk} \) depend only on the usage of routes using the associated link.

**B. Stability and instability**

It is well known (see e.g. [1],[3]) that the distance to any equilibrium point is non-increasing along the trajectories.
of the (sub)gradient method \(^{(1)},(2)\), which we state in the lemma below.

**Lemma 5.** Let \( \varphi \in C^2 \) be concave-convex, and \( \bar{z} = (\bar{x}, \bar{y}) \) be an equilibrium point of the subgradient method \(^{(2)}\). Then for any trajectory \( \mathbf{z}(t) = (x(t), y(t)) \) the distance \( |x(t) - \bar{z}|^2 \) is non-increasing in time.

This immediately implies the stability of such equilibrium points, and this stability carries over to the congestion control dynamics \(^{(5)}\). However, asymptotic stability is known to not hold in general for functions \( \varphi \) that are not strictly-concave, which is the case here.

We will now give examples of this lack of asymptotic stability. A more detailed discussion of limiting solutions of the gradient method may be found in \(^{[6]}\), see also \(^{[5]}\) for a detailed study in a particular case which has many similarities to the situation considered here.

For simplicity we shall consider the unconstrained dynamics given by

\[
\dot{x}_j = U_i' \left( \sum_{k \sim r_j} x_k \right) - \sum_{l_k \sim r_j} y_k \tag{6}
\]

\[
\dot{y}_k = \sum_{l_k \sim r_j} x_j - C_k
\]

where, as before, \( s_i \sim x_j \) in the first equation. We will show, that under the algebraic condition

\[
\exists u \in \ker(H) \setminus \{0\}, \lambda > 0 \text{ such that } A^T A u = \lambda u, \tag{7}
\]

which we note holds in many networks (e.g. that shown in Figure 1), the dynamics \(^{(6)}\) have a conserved quantity.

**Lemma 6.** Let \(^{(1)}\) hold and \( U_i \in C^2 \) be strictly concave and strictly increasing. Let \( (\bar{x}, \bar{y}) \) be a saddle point of \(^{(4)}\) and define

\[
W(x, y; \bar{x}) = \frac{1}{2} \| (x - \bar{x})^T u \|^2 + \frac{1}{\lambda} \| y^T A u \|^2, \tag{8}
\]

then \( W \) is constant along trajectories of the unconstrained dynamics \(^{(6)}\).

**Proof.** The derivative along trajectories of \(^{(6)}\) is given by

\[
\dot{W} = W_x \varphi_x - W_y \varphi_y
\]

\[
= (x^T - \bar{x})^T u^T \varphi_x - \frac{1}{\lambda} y^T A u A^T (C - Ax) \tag{9}
\]

where \( \varphi \) is given by \(^{(4)}\). As \( u \in \ker(H) \) and the first term in \(^{(4)}\) is a function of \( H x, u^T \varphi_x = -u^T A^T y. \) Thus,

\[
\dot{W} = (x^T - \bar{x})^T u^T A^T y + \frac{1}{\lambda} x^T A u A^T y
\]

\[
+ (x^T - \bar{x})^T u^T A^T y - \frac{1}{\lambda} C x^T A u A^T y. \tag{10}
\]

The first line vanishes because \( u \) is an eigenvector, and the second line vanishes as \( C = A \bar{x} \) is implied by \( \varphi_y \) vanishing at the saddle point \( (\bar{x}, \bar{y}) \).

This means that under the condition \(^{(7)}\), only trajectories that have \( W \) initially zero for some saddle point can converge, and this happens only for a very small subset of initial conditions.

The existence of a conserved quantity also makes the system susceptible to perturbative noise, which we will now illustrate. Consider the addition of a driving white noise to the dynamics \(^{(6)}\). This gives the following stochastic differential equations

\[
dx_j(t) = \left( U_i' \left( \sum_{k \sim r_j} x_k \right) - \sum_{l_k \sim r_j} y_k \right) dt + \sigma_j^x dW_j^x(t)
\]

\[
dy_k(t) = \left( \sum_{l_k \sim r_j} x_j - C_k \right) dt + \sigma_k^y dW_k^y(t)
\]

where \( s_i \sim x_j \) in the equation for \( x_j, \sigma_j^x, \sigma_k^y > 0 \) are constant infinitesimal variances, and \( W_j^x(t), W_k^y(t) \) are mutually independent standard Brownian motions. Then, under the same algebraic criterion, we show that the variance of the solution to \(^{(11)}\) grows without bound.

**Proposition 7.** Let \( U_i \in C^2 \) be strictly concave and strictly increasing. Consider the unconstrained noisy dynamics \(^{(11)}\) for some strictly positive infinitesimal variances \( \sigma_j^x, \sigma_k^y \). Assume that \(^{(7)}\) holds. Then, for any initial condition, the variance of the solution to \(^{(11)}\) tends to infinity as \( t \to \infty \), in that

\[
\mathbb{E}(|x(t)|^2 + |y(t)|^2) \to \infty \text{ as } t \to \infty. \tag{12}
\]

where \( \mathbb{E} \) denotes the expectation operator.

**Proof.** Consider the conserved quantity \( W(x, y; \bar{x}) \) given by \(^{(8)}\) for some saddle point \((\bar{x}, \bar{y})\). Note that \( 0 \leq W(x, y; \bar{x}) \leq C(|x|^2 + |y|^2 + 1) \) for some constant \( C \), so it is sufficient to show that \( \mathbb{E} W(x(t), y(t); \bar{x}) \to \infty \) as \( t \to \infty \).

Applying Itô's lemma and taking expectations, we have

\[
\frac{d}{dt} \mathbb{E}(W(x(t), y(t); \bar{x})) = \mathbb{E}(W_x^T \varphi_x - W_y^T \varphi_y)
\]

\[
+ \frac{1}{2} \mathbb{E}(\text{Tr}(\Sigma^T W_{xx} \Sigma)) \tag{13}
\]

where \( \varphi \) is defined by \(^{(4)}\) and \( \Sigma \) is the diffusion matrix given by \( \text{diag}(\sigma_1^x, \ldots, \sigma_r^x, \sigma_1^y, \ldots, \sigma_l^y) \). A simple computation shows that the second term is a strictly positive constant. The first term is the expectation of the derivative of \( W \) along the deterministic flow \(^{(6)}\), so vanishes by Lemma 6. Hence \( \mathbb{E}(W(x(t), y(t); \bar{x})) \) increases linearly with time. \( \square \)

### IV. Modified Dynamics

In this section we present a modification of the dynamics \(^{(5)}\), that, while still fully localised, give guaranteed convergence to an optimal solution of \(^{(3)}\).

We define a modified optimisation problem

\[
\max_{x \geq 0, x' \in \mathbb{R}^n} \sum_{s_i \in S} U_i \left( \sum_{r_j \sim s_i} x_j \right) - \frac{1}{2} \sum_{r_k} \kappa_k |x'_k - x_k|^2 \tag{14}
\]

where \( x' \in \mathbb{R}^n \) is an additional vector to be optimised over, and \( \kappa_k > 0 \) are arbitrary constants. It is important to note
that this has the same optimal \( x \) points as \( \mathbf{9} \). This gives rise to a modified Lagrangian

\[
\varphi'(x', x, y) = \sum_{i} U_i \left( \sum_{j} x_j \right) + y^T (C - Ax) - \frac{1}{2} \sum_{k} \kappa_k |x_k' - x_k|^2.
\]

(15)

The new dynamics are given by the following subgradient method.

\[
\dot{x}_j = \left[ U_i \left( \sum_{i} x_i \right) - \sum_{k} y_k + \kappa_j (x_j' - x_j) \right]_{x_j}^{+} \tag{16}
\]

\[
\dot{y}_k = \left[ \sum_{i} x_i - C_k \right]_{y_k}^{+}.
\]

The general form of the modified dynamics is given by the following subgradient method.

To prove Theorem 9 we will use the following convergence criterion for the subgradient method \( \mathbf{9} \). This result is part of an ongoing study by the authors into the convergence properties of the subgradient method in its general form on convex domains. Due to page constraints the proof has been omitted, and may be found in an extended version of this manuscript \( \mathbf{7} \).

Proposition 11. Let \( C^2 \supseteq \varphi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) be concave-convex, and let \( I \subseteq \{1, \ldots, n\} \) and \( J \subseteq \{1, \ldots, m\} \). Then the subgradient method \( \mathbf{2} \) is globally asymptotically stable if, for any \( I' \subseteq I \) and \( J' \subseteq J \) such that \( \varphi' \) has a saddle point, the unconstrained dynamics \( \mathbf{1} \) obtained from \( \varphi' \) are globally asymptotically stable, where \( \varphi' \) is defined by \( \varphi'(x, y) = \varphi(x', y') \) and

\[
x_i' = \begin{cases} 0 & \text{if } i \in I', \\
x_i & \text{if } i \notin I'.
\end{cases}
\]

\[
y_j' = \begin{cases} 0 & \text{if } j \in J', \\
y_j & \text{if } j \notin J'.
\end{cases}
\]

(18)

The effect of this result is that to prove convergence of the constrained dynamics \( \mathbf{2} \), it is sufficient to prove convergence of the unconstrained dynamics \( \mathbf{1} \) for \( \varphi \) with an arbitrary subset of the coordinates set to zero. In the case of multi-path routing, fixing coordinates to zero corresponds to changing the network topology or the running capacity constraints. Following this strategy, we prove convergence of the unconstrained dynamics.

Lemma 12. Let \( U_i \in C^2 \) be strictly concave and strictly increasing. Then the unconstrained dynamics obtained from \( \mathbf{16} \) by removing the positivity projections are globally asymptotically stable.

Proof. Let \( \bar{z} = (\bar{x}, \bar{y}) \) be a saddle point of \( \varphi \) defined by \( \mathbf{4} \). First consider a trajectory \( z(t) \) of the unmodified dynamics \( \mathbf{6} \). By taking \( \frac{1}{2} \| z(t) - \bar{z} \|^2 \) and using Lemma 5, we have

\[
\left[ x - \bar{x} \right]^T \left[ \varphi_x \right] \leq 0,
\]

(19)

and as this holds for all initial conditions \( z(0) \), the inequality \( \mathbf{19} \) holds for all \( (x, y) \).

Next consider the unconstrained modified dynamics, for which \( \dot{z}' = (\dot{x}', \dot{y}') \) is an equilibrium point. By the classical LaSalle’s theorem (see e.g. \[10]\), using \( \frac{1}{2} \| z'(t) - \bar{z}' \|^2 \) as the Lyapunov like function, (which is non-increasing along trajectories by Lemma 5), any limiting solution \( z'(t) = (x(t), x'(t), y(t)) \) satisfies \( \frac{1}{2} \| z'(t) - \bar{z}' \|^2 = 0 \), which gives the equality

\[
0 = \left[ x - \bar{x} \right]^T \left[ K \right] \left[ x \right] + \left[ y - \bar{y} \right]^T \left[ -\varphi_y \right]
\]

where \( K = \text{diag}(\kappa_1, \ldots, \kappa_n) \). Both terms are non-positive. The first as it is equal to \( -(x - x')^T K (x - x') \) and the second by \( \mathbf{19} \) deduced above. Therefore both terms vanish and \( x'(t) = x(t) \) for all \( t \). By the ODE for \( x' \) we deduce that
x' is constant, and hence x is constant. Then the ODE for y implies that \( y \) is constant, and as \( |y(t)|^2 \leq |z'(0)|^2 < \infty \) for \( t \geq 0 \), \( y \) is constant. Hence \( (x, x', y) \) is an equilibrium point and the dynamics are globally asymptotically stable.

From this we now prove the convergence of the constrained dynamics (16) using the criterion Proposition 11.

**Proof of Theorem 9.** By Proposition 11 it suffices to consider the unconstrained dynamics with some arbitrary set of coordinates fixed at zero. Fixing \( x_j = 0 \) corresponds in the optimisation problem to removing the use of the route \( r_j \). Fixing \( y_k = 0 \) corresponds to removing the constraint \( \sum_{r_i} \omega_{l_i} x_j \leq C_k \). Thus each corresponds to a modification of the network topology or constraints, and as the proof of Lemma 12 above applies to all such topologies and constraints, it applies here also. (Note that Proposition 11 allows us to assume that the newly obtained topology and constraints still give rise to at least one saddle point.) Finally, as discussed above, the equilibrium points of the modified dynamics (16) correspond to the equilibrium points of the original dynamics (5), and these are the solutions of the original optimisation problem (3).

V. NUMERICAL RESULTS

In this section we present numerical simulations to illustrate our analytic results. We consider the two networks in Figure 1 and Figure 4.

In Figure 2 and Figure 3 we use the network in Figure 1 with capacities all set to 1. The utility functions were chosen as \( \log(1+x) \) and \( 1 - e^{-x} \) for the sources at 1 and 2 respectively. The parameters \( \kappa_j \) were all set to 1. This network satisfies the conditions of Lemma 6 and this is apparent in the oscillating modes of the unmodified dynamics (5), shown in Figure 2 that do not decay. However, when we apply the modified dynamics (16) to this network, we obtain the rapid convergence to the equilibrium shown in Figure 3.

In Figure 5 and Figure 6 we use the network in Figure 4. We take the utility function as \( \log(1+x) \), and the capacities all set to 0.5. The parameters \( \kappa_j \) were all set to 1. On this network the original dynamics (5) converge to equilibrium, shown in Figure 5, but there is transient oscillatory behaviour. When we instead implement the modified dynamics (16), we see an improved performance with more rapid convergence and damping of the oscillations.

Fig. 1. A first example network. Sources at 1 and 2 transmit to the destinations 4 and 3 respectively. Each has a choice of two routes. Routes associated with the source at 1 are dotted lines, while those associated with the source at 2 are solid lines.

Fig. 2. The unmodified dynamics (5) running on the network given in Figure 1 with all capacities set to 1 and the utility functions are \( \log(1+x) \) and \( 1 - e^{-x} \) for the sources at 1 and 2 respectively. In this network the condition (7) holds, and there is oscillatory behaviour which does not decay.

Fig. 3. The modified dynamics (16) running on the network given in Figure 1 with all capacities set to 1 and \( \kappa_j = 1 \) for all j. The utility functions are \( \log(1+x) \) and \( 1 - e^{-x} \) for the sources at 1 and 2 respectively. In this network the condition (7) holds, but the modification of the dynamics causes rapid convergence to equilibrium.

Fig. 4. A second example network. A single source at 1 transmits to the destination 7. It has a choice of two routes.
In this paper we considered the problem of multi-path routing in a communication network where a prescribed number of routes is available per source/destination pair. We proved that a classical implementation of primal-dual gradient dynamics is susceptible to noise amplification becoming unbounded, in some network topologies, with arbitrarily small noise. Then we showed that, by the addition of higher order dynamics, convergence may be achieved to an optimal solution without making use of a relaxation of the problem or introducing additional information exchange. Improvements in performance have also been illustrated by means of simulations. Potential future work includes quantifying these improvements by means of a robustness analysis in the presence of delays.

VI. CONCLUSIONS

In this paper we considered the problem of multi-path routing in a communication network where a prescribed number of routes is available per source/destination pair. We proved that a classical implementation of primal-dual gradient dynamics is susceptible to noise amplification becoming unbounded, in some network topologies, with arbitrarily small noise. Then we showed that, by the addition of higher order dynamics, convergence may be achieved to an optimal solution without making use of a relaxation of the problem or introducing additional information exchange. Improvements in performance have also been illustrated by means of simulations. Potential future work includes quantifying these improvements by means of a robustness analysis in the presence of delays.

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